Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337

TMA 01 2020J

Covers Book A **Cut-off date 2 December 2020**

You can submit this TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on it, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' area of the M337 website.

Your work should be written in good mathematical style, as demonstrated by the example and exercise solutions in the study units. You should explain your solutions carefully, using appropriate notation and terminology, and write in sentences. As usual, you should simplify algebraic answers where possible.

In the wording of the questions:

- *write down* or *state* means 'write down without justification'
- *find*, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer
- *prove*, *show* or *deduce* means that you should carefully justify each step of your solution.

Make sure you reference any significant result from the module materials that you use, and check that all the conditions of the result are satisfied.

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4.1

Question 1 (Unit A1) – 25 marks

(a) Let $w_1 = 2\sqrt{3} + 2i$ and $w_2 = -2\sqrt{3} + 2i$.

(i) Evaluate each of the following numbers, giving your answers in Cartesian form:

$$w_1 w_2, \quad w_1^2 - w_2^2, \quad \frac{1}{w_1}.$$ *[3]*

(ii) Express w_2 in polar form, and hence determine all the fourth roots of w_2, giving your answers in polar form. *[4]*

(iii) Solve the equation

$$z^4 - 4iz^2 - 4 = 0,$$

giving your answers in Cartesian form. *[6]*

(b) Let

$$A = \{z : -\pi/2 \leq \operatorname{Arg}(z+1) < \pi/4\},$$
$$B = \{z : |z+i| \geq 1\}.$$

Sketch the sets A, B and $A \cap B$ on separate diagrams using the sketching conventions from Subsection 4.2 of Unit A1. *[6]*

(c) (i) Prove that

$$|z^3 + z^2 - 3i| \geq 1 \quad \text{and} \quad |z^4 + 3 - i| \leq 20,$$

for $|z| = 2$. *[4]*

(ii) Find a positive number M such that

$$\left| \frac{z^3 + z^2 - 3i}{z^4 + 3 - i} \right| \geq M, \quad \text{for } |z| = 2.$$ *[2]*

Question 2 (Unit A2) – 25 marks

(a) Let

$$f_1(z) = \tfrac{1}{2}z, \quad f_2(z) = -iz \quad \text{and} \quad f_3(z) = z + 2i,$$

and let $f = f_1 \circ f_2 \circ f_3$.

(i) Describe the geometric effect on the complex plane of each of the functions f_1, f_2 and f_3. *[3]*

(ii) Determine the rule of the function f, simplifying your answer as far as possible. *[1]*

(iii) Let Γ be the path with parametrisation

$$\gamma(t) = e^{it} - i \quad (t \in [0, \pi]).$$

Sketch Γ, indicating the direction of increasing t, and identify its initial and final points in Cartesian form. *[2]*

(iv) Sketch the path $f(\Gamma)$ by applying the geometric transformations f_3, f_2 and f_1, in that order, showing the effect of each transformation in turn on separate diagrams. Indicate the direction of $f(\Gamma)$, and identify its initial and final points in Cartesian form. *[3]*

(v) Write down the standard parametrisation of $f(\Gamma)$. *[2]*

(b) Let

$$f(z) = \frac{z^2 - 1}{z^2 + 1}, \quad g(z) = \mathrm{Log}(3z) \quad \text{and} \quad h(z) = \cosh z.$$

(i) Write down the domain of each of the functions f, g and h. [3]

(ii) Evaluate each of the function values $g(-1 - i)$ and $h(\pi + \pi i)$, giving your answers in Cartesian form. [4]

(iii) Determine which of the three functions f, g and h has an inverse function.

For each function that does have an inverse function, find the domain and rule of the inverse function.

For each function that does not have an inverse function, demonstrate why it does not have an inverse function. [7]

Question 3 (Unit A3) – 25 marks

(a) Determine whether each of the following sequences converges, and if it does, then state the limit.

(i) $z_n = \dfrac{(1 + 2i)^n - (1 + i)^n}{(1 + 2i)^n + (1 - i)^n}, \quad n = 1, 2, \ldots$ [4]

(ii) $z_n = \exp(n\pi i/3), \quad n = 1, 2, \ldots$ [3]

(b) Prove that each of the following functions is continuous, stating any rules that you use.

(i) $f(z) = \dfrac{z^2}{\cosh z} \quad (|z| < 1)$ [3]

(ii) $f(z) = z^i \quad (\mathrm{Re}\, z > 0)$ [1]

(iii) $f(z) = i^z \quad (z \in \mathbb{C})$ [2]

(c) Let $S = \{z : |\mathrm{Re}\, z| > 1\} \cup \{z : |\mathrm{Im}\, z| > 1\}$ and let

$$A = S \cap \{z : |z| < 2\},$$
$$B = A \cup \partial A,$$
$$C = \mathbb{C} - B.$$

(i) Sketch the sets A, B and C on separate diagrams. [3]

(ii) State whether or not each of the sets A, B and C is a region. Justify your answers briefly. [3]

(iii) State whether or not each of the sets A, B and C is compact. Justify your answers briefly. [3]

(iv) Using your answer to part (c)(iii), or otherwise, prove that the function

$$f(z) = \frac{1}{7z^7 - 1}$$

is bounded on B.

(You are *not* asked to find an upper bound for f on B.) [3]

Question 4 (Unit A4) – 25 marks

(a) Find the derivative of the function

$$f(z) = \frac{z^3 + z}{z^4 + 1},$$

and specify the domain of this derivative. [*3*]

(b) Use the Cauchy–Riemann Theorem and the Cauchy–Riemann Converse Theorem to determine all points of the complex plane at which the function

$$f(z) = \overline{z}(z + i)$$

is differentiable.

Find the derivative of f at each point at which it is differentiable. [*10*]

(c) Let $f(z) = \operatorname{Log}(z - i) - z$.

(i) Determine all the points of the complex plane at which f is differentiable, and determine all points of the complex plane at which f is conformal. [*4*]

(ii) Show that the paths

$$\Gamma_1 : \gamma_1(t) = 1 + 2\cos t + i \sin t \quad (t \in [-\pi, \pi]),$$
$$\Gamma_2 : \gamma_2(t) = t + i(t - 2) \quad (t \in \mathbb{R})$$

are smooth and meet at the point $1 - i$. [*4*]

(iii) Determine the angle from the path $f(\Gamma_1)$ to the path $f(\Gamma_2)$ at the point $f(1 - i)$. [*4*]